History of Mathematics

Hist1

Sources for Group Project A

Tangent and Max-Min Methods in the 17th Century before Newton and Leibniz

De følgende kilder er hentet fra tre kildesamlinger, nemlig:
Ronald Calinger: *Classics of Mathematics*, Prentice Hall, Engelwood Cliffs, 1995. (kilde 1).
Dirk J. Struik: *A Sourcebook in Mathematics*, Harvard University Press, Cambridge Mass. 1969. (kilde 2)
John Fauvel & Jeremy Gray: *The History of Mathematics. A Reader*, Macmillan, London 1987. (kilde 3 og 4)

Kilde 1

From "On a Method for the Evaluation of Maxima and Minima"[1*]

(Fermat obtained a general method to find the extrema of a given function. His algorithm was subsequently developed into the method of the "characteristic triangle," *dx, dy,* and *ds.*)

– PIERRE DE FERMAT

The whole theory of evaluation of maxima and minima presupposes two unknown quantities and the following rule:

Let a be any unknown of the problem (which is in one, two, or three dimensions, depending on the formulation of the problem). Let us indicate the maximum or minimum by a in terms which could be of any degree. We shall now replace the original unknown a by $a + e$ and we shall express thus the maximum or minimum quantity in terms of a and e involving any degree. We shall adequate [*adégaler*], to use Diophantus' term,[2] the two expressions of the maximum or minimum quantity and we shall take out their common terms. Now it turns out that both sides will contain terms in e or its powers. We shall divide all terms by e, or by a higher power of e, so that e will be completely removed from at least one of the terms. We suppress then all the terms in which e or one of its powers will still appear, and we shall equate the others; or, if one of the expressions vanishes, we shall equate, which is the same thing, the positive and negative terms. The solution of this last equation will yield the value of a, which will lead to the maximum or minimum, by using again the original expression.

Here is an example:

To divide the segment AC [Fig. 70.1] *at E so that AE × EC may be a maximum.*

Figure 70.1

We write $AC = b$; let a be one of the segments, so that the other will be $b - a$, and the product, the maximum of which is to be found, will be $ba - a^2$. Let now $a + e$ be the first segment of b; the second will be $b - a - e$, and the product of the segments, $ba - a^2 + be - 2ae - e^2$; this must be adequated with the preceding: $ba - a^2$. Suppressing common terms: $be \sim 2ae + e^2$. Suppressing e: $b = 2a$.[3] To solve the problem we must consequently take the half of b.

We can hardly expect a more general method.

On the Tangents of Curves

We use the preceding method in order to find the tangent at a given point of a curve.

Let us consider, for example, the parabola BDN [Fig. 70.2] with vertex D and of diameter DC; let B be a point on it at which the line BE is to be drawn tangent to the parabola and intersecting the diameter at E.

We choose on the segment BE a point O at which we draw the ordinate OI; also we construct

* SOURCE: This translation from the *Oeuvres de Fermat*, edited by P. Tannery and C. Henry, appears in D. J. Struik, ed., *A Source Book in Mathematics, 1200–1800* (1969), 223–27. It is reprinted with permission of Harvard University Press, Copyright © 1969 by the President and Fellows of Harvard College.

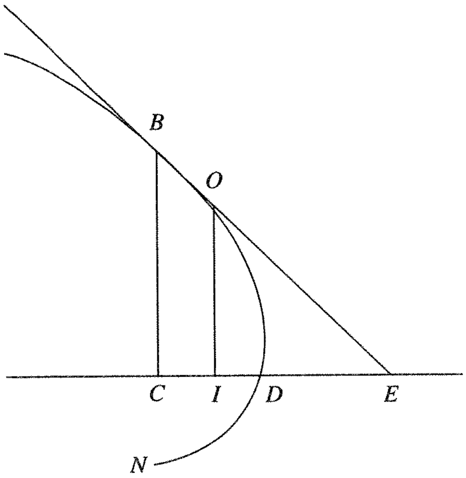


Figure 70.2

the ordinate BC of the point B. We have then: $CD/DI > BC^2/OI^2$, since the point O is exterior to the parabola. But $BC^2/OI^2 = CE^2/IE^2$, in view of the similarity of triangles. Hence $CD/DI > CE^2/IE^2$.

Now the point B is given, consequently the ordinate BC, consequently the point C, hence also CD. Let $CD = d$ be this given quantity. Put $CE = a$ and $CI = e$; we obtain

$$\frac{d}{d-e} > \frac{a^2}{a^2+e^2-2ae}. \quad [4]$$

Removing the fractions:

$$da^2 + de^2 - 2dae > da^2 - a^2e.$$

Let us then adequate, following the preceding method; by taking out the common terms we find:

$$de^2 - 2dae \sim -a^2e,$$

or, which is the same,

$$de^2 + a^2e \sim 2dae.$$

Let us divide all terms by e:

$$de + a^2 \sim 2da.$$

On taking out de, there remains $a^2 = 2da$, consequently $a = 2d$.

Thus we have proved that CE is the double of CD—which is the result.

This method never fails and could be extended to a number of beautiful problems; with its aid, we have found the centers of gravity of figures bounded by straight lines or curves, as well as those of solids, and a number of other results which we may treat elsewhere if we have time to do so.

I have previously discussed at length with M. de Roberval[5] the quadrature of areas bounded by curves and straight lines as well as the ratio that the solids which they generate have to the cones of the same base and the same height.

[Now follows the second illustration of Fermat's "e-method," where Fermat's e = Newton's o = Leibniz' dx.[6]]

Center of Gravity of Paraboloid of Revolution, Using the Same Method[7]

Let $CBAV$ (Fig. 70.3) be a paraboloid of revolution, having for its axis IA and for its base a circle of diameter CIV. Let us find its center of gravity by using the same method which we applied for maxima and minima and for the tangents of curves; let us illustrate, with new examples and with new and brilliant applications of this method, how wrong those are who believe that it may fail.

In order to carry out this analysis, we write $IA = b$. Let O be the center of gravity, and a the unknown length of the segment AO; we intersect

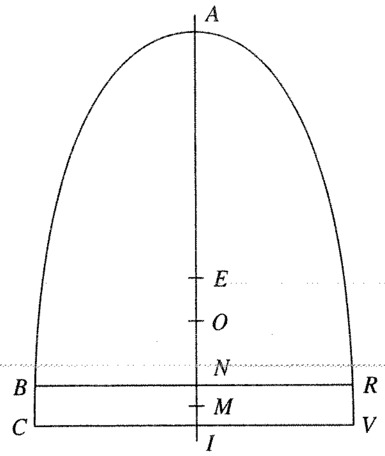


Figure 70.3

the axis IA by any plane BN and put $IN = e$, so that $NA = b - e$.

It is clear that in this figure and in similar ones (parabolas and paraboloids) the centers of gravity of segments cut off by parallels to the base divide the axis in a constant proportion (indeed, the argument of Archimedes can be extended by similar reasoning from the case of a parabola to all parabolas and paraboloids of revolution[8]). Then the center of gravity of the segment of which NA is the axis and BN the radius of the base will divide AN at a point E such that $NA/AE = IA/AO$, or, in formula, $b/a = (b - e)/AE$.

The portion of the axis will then be $AE = (ba - ae)/b$ and the interval between the two centers of gravity, $OE = ae/b$.

Let M be the center of gravity of the remaining part $CBRV$; it must necessarily fall between the points N, I, inside the figure, in view of Archimedes' postulate 9 in *On the equilibrium of planes*, since $CBRV$ is a figure completely concave in the same direction.[9]

But

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{OE}{OM},$$

since O is the center of gravity of the whole figure CAV and E and M are those of the parts.

Now in the paraboloid of Archimedes,

$$\frac{\text{Part } CAV}{\text{Part } BAR} = \frac{IA^2}{NA^2} = \frac{b^2}{b^2 + e^2 - 2be};$$

hence by dividing,

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{2be - e^2}{b^2 + e^2 - 2be}.$$

But we have proved that

$$\frac{\text{Part } CBRV}{\text{Part } BAR} = \frac{OE}{OM}.$$

Then in formulas,

$$\frac{2be - e^2}{b^2 + e^2 - 2be} = \frac{OE\,(= ae/b)}{OM};$$

hence

$$OM = \frac{b^2ae + ae^3 - 2bae^2}{2b^2a - be^2}.$$

From what has been established we see that the point M falls between points N and I; thus $OM < OI$; now, in formula, $OI = b - a$. The question is then prepared from our method, and we may write

$$b - a \sim \frac{b^2ae + ae^3 - 2bae^2}{2b^2e - be^2}.$$

Multiplying both sides by the denominator and dividing by e:

$$2b^3 - 2b^2a - b^2e + bae \sim b^2a + ae^2 - 2bae.$$

Since there are no common terms, let us take out those in which e occurs and let us equate the others:

$$2b^3 - 2b^2a = b^2a, \quad \text{hence} \quad 3a = 2b.$$

Consequently

$$\frac{IA}{AO} = \frac{3}{2}, \quad \text{and} \quad \frac{AO}{OI} = \frac{2}{1},$$

and this was to be proved.[10]

The same method applies to the centers of gravity of all the parabolas ad infinitum as well as those of paraboloids of revolution. I do not have time to indicate, for example, how to look for the center of gravity in our paraboloid obtained by revolution about the ordinate;[11] it will be sufficient to say that, in this conoid, the center of gravity divides the axis into two segments in the ratio 11/5.

Struik's Notes

1. This paper was sent by Fermat to Father Marin Mersenne, who forwarded it to Descartes. Descartes received it in January 1638. It became the subject of a polemic discussion between him and Fermat (*Oeuvres,* I, 133). On Mersenne, see Selection I.6, note 1, of Struik.
2. See Selection IV.7, note 5 [of Struik].
3. Our notation is modern. For instance, we have written (following the French translation in *Oeuvres,* III, 122) $be \sim 2ae + e^2$, Fermat wrote: B in E adaequabitur A in E bis + Eq (Eq standing for E quadratum). The symbol $\sim$ is used for "adequates."

[4.] Fermat wrote: D ad $D - E$ habebit majorem proportionem quam $Aq.$ ad $Aq. + Eq. - A$ in E bis (D will have to $D - E$ a larger ratio than A^2 to $A^2 + E^2 - 2AE$).

5. See the letters from Fermat to Roberval, written in 1636 (*Oeuvres,* III, 292–94, 296–97).
6. The gist of this method is that we change the variable x in $f(x)$ to $x + e$, e small. Since $f(x)$ is stationary near a maximum or minimum (Kepler's remark), $f(x + e) - f(x)$ goes to zero faster than e does. Hence, if we divide by e, we obtain an expression that yields the required values for x if we let e be zero. The legitimacy of this procedure remained, as we shall see, a subject of sharp controversy for many years. Now we see in it a first approach to the modern formula:

$$f'(x) = \lim_{e \to 0} \frac{f(x + e) - f(x)}{e},$$

introduced by Cauchy (1820–21).

7. This paper seems to have been sent in a letter to Mersenne written in April 1638, for transmission to Roberval. Mersenne reported its contents to Descartes. Fermat used the term "parabolic conoid" for what we call "paraboloid of revolution."
8. "All parabolas" means "parabolas of higher order," $y = kx^n$, $n > 2$. The reference is to Archimedes' *On Floating Bodies,* II, Prop. 2 and following; see T. L. Heath, *The Works of Archimedes* (Cambridge University Press, Cambridge, England, 1897; reprint, Dover, New York), 264ff.
9. This is postulate 7 in the modern Heiberg edition, and is translated in Heath, p. 190, as follows: "In any figure whose perimeter is concave in (one and) the same direction the center of gravity must be within the figure." (On the term "concave in the same direction," see Heath, p. 2.)
10. These relations were known to Archimedes (see note 8). But Fermat solved this problem on centers of gravity, hence a problem in the integral calculus, with what we might call an application of the principle of virtual variations.
11. Here ACI of Fig. 70.3 is rotated about CI.

Kilde 2 ROBERVAL. THE CYCLOID

To Generate the Cycloid. Let the diameter AB [Fig. 1] of the circle $AEGB$ move along the tangent AC, always remaining parallel to its original position, until it takes the position CD, and let AC be equal to the semicircle AGB. At the same time, let the point A move on the semicircle AGB, in such a way that the speed of AB along AC may be equal to the speed of A along the semicircle AGB. Then, when AB has reached the position CD, the point A will have reached the position D. The point A is carried along by two motions—its own on the semicircle $AEGB$, and that of the diameter along AC. The path of the point A, due to these two motions, is the half cycloid $A \cdots D$, the second half being symmetrical with the first.

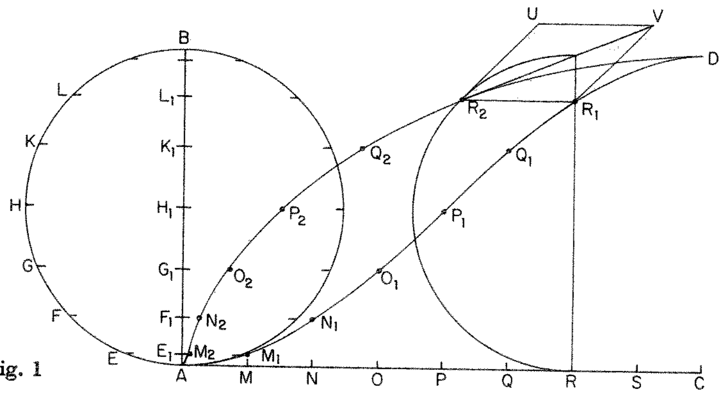


Fig. 1

Proposition 3. *To construct a tangent to the cycloid.*

Construction. Let R_2 be the given point at which the tangent is to be drawn. Draw R_2R_1 parallel to AC. Draw R_2U tangent to the generating circle RR_2 and make $R_2U = R_2R_1$. Complete the parallelogram R_2UVR_1, and draw the diagonal R_2V. Then R_2V is the required tangent.

Proof. The direction of the motion of the point R_2 which is due to the motion of AB along AC is R_2R_1; the direction of the motion of the point R_2 which is due to the motion of the point A on the circumference is R_2U, and since these motions are always equal, it follows that R_2R_1 must equal R_2U. Therefore R_2V is the tangent to the cycloid at R_2, since it is the resultant of the two motions.

Addendum. If, instead of being equal, the magnitudes of the two motions had been in some other ratio, the parallelogram would have been constructed with its sides in that ratio.[4]

[2] The "companion of the cycloid" is a sine curve. If AC is taken as the X-axis, AB as the Y-axis, its equation is, in our notation, $y = 1 - \cos x$.

[3] When arc $AE = \varphi$ [radius $R = 1$], then the equation of the cycloid is $x = \varphi - \sin \varphi$, $y = 1 - \cos \varphi$, and the area $AM_2DM_1A = \int_0^\pi (\varphi - \varphi + \sin \varphi)^2\, dy = \int_0^\pi \sin^2 \varphi \, d\varphi = \pi/2$.

[4] The tangent construction uses kinematic concepts and is related to the method of Archimedes in his book *On spirals*; see T. L. Heath, *The works of Archimedes* (Cambridge University Press, Cambridge, England, 1897; reprint, Dover, New York), 151ff, esp. Props. 16–20. While Roberval used Greek methods to find tangents, his contemporary Fermat was laying the foundations of the present method, based on the derivative (see Selection IV.8).

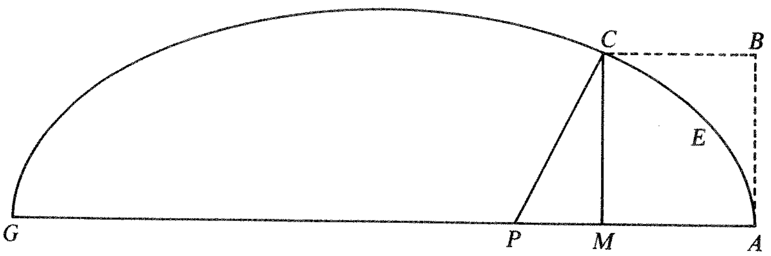
Kilde 3

DESCARTES

11.A9 The method of normals

[1] The angle formed by two intersecting curves can be as easily measured as the angle between two straight lines, provided that a straight line can be drawn making right angles with one of these curves at its point of intersection with the other. This is my reason for believing that I shall have given here a sufficient introduction to the study of curves when I have given a general method of drawing a straight line making right angles with a curve at an arbitrarily chosen point upon it. And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know.

[2] Let CE be the curved line. It is desired to draw a straight line at right angles to it, through the point C. I suppose the problem to have been solved, and that the sought-for line is CP, which I prolong to the point P where it meets the straight line GA. (GA is the line to whose points all those of CE are referred; so that putting MA or CB equal to y, and CM or BA equal to x, I have some equation showing the relation between x and



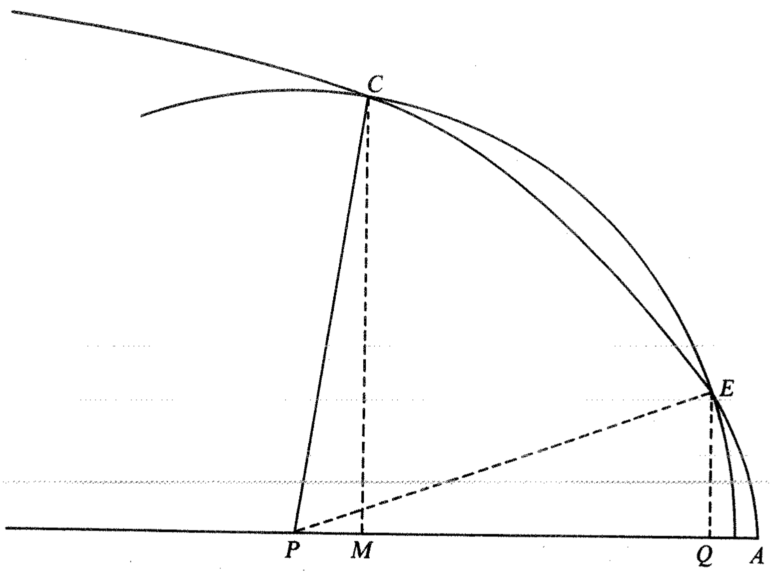
y.) Then I put $PC = s$, and $PA = v$, whence $PM = v - y$. Since the triangle PMC is right-angled, the square on the hypotenuse s^2 is equal to $x^2 + v^2 - 2vy + y^2$, the sum of the squares on the two sides. That is to say, $x = \sqrt{s^2 - v^2 + 2vy - y^2}$ or equally $y = v + \sqrt{s^2 - x^2}$. By this means I can get rid of one of the two unknown quantities x or y from the equation relating the points of the curve CE to those of the straight line GA. This is easily done by putting throughout $\sqrt{s^2 - v^2 + 2vy - y^2}$ in place of x, the square of this in the place of x^2, its cube in place of x^3, and so on. That is if it's x I want to get rid of; or if it's y, I put in its place $x + \sqrt{s^2 - x^2}$, and its square or cube, etc., in place of y^2, y^3 etc. After this process there always remains an equation in only one unknown quantity, x or y.

[3] For example, if CE is an Ellipse, MA the segment of its diameter on which CM is ordinate, and which has r for its *latus rectum* and q its major axis then by Book I Proposition 13 of Apollonius we have $x^2 = ry - ry^2/q$. Getting rid of x^2 from this gives $s^2 - v^2 + 2vy - y^2 = ry - ry^2/q$, or

$$y^2 + \frac{qry - 2qvy + qv^2 - qs^2}{q - r}$$

equals nothing. For it is better here to consider the whole together in this way, than as one part equal to the other. [...]

[4] Such an equation having been found it is to be used, not to determine x, y, or z, which are known, since the point C is given, but to find v or s, which determine the required point P. With this in view, observe that if the point P fulfills the required conditions, the circle about P as centre and passing through the point C will touch but not cut the curve CE; but if this point P be ever so little nearer to or farther from A than



it should be, this circle must cut the curve not only at C but also in another point. Now if this circle cuts CE, the equation involving x and y as unknown quantities (supposing PA and PC known) must have two unequal roots. Suppose, for example, that the circle cuts the curve in the points C and E. Draw EQ parallel to CM. Then x and y may be used to represent EQ and QA respectively in just the same way as they were used to represent CM and MA; since PE is equal to PC (being radii of the same circle), if we seek EQ and QA (supposing PE and PA given) we shall get the same equation that we should obtain by seeking CM and MA (supposing PC and PA given). It follows that the value of x, or y, or any other such quantity, will be two-fold in this equation, that is, the equation will have two unequal roots. If the value of x be required, one of these roots will be CM and the other EQ; while if y be required, one root will be MA and the other QA. It is true that if E is not on the same side of the curve as C, only one of these will be a true root, the other being drawn in the opposite direction, or less than nothing. The nearer together the points C and E are taken however, the less difference there is between the roots; and when the points coincide, the roots are exactly equal, that is to say, the circle through C will touch the curve CE at the point C without cutting it.

[5] Furthermore, it is to be observed that when an equation has two equal roots, its left-hand member must be similar in form to the expression obtained by multiplying by itself the difference between the unknown quantity and a known quantity equal to it; and then, if the resulting expression is not of as high a degree as the original equation, multiplying it by another expression which will make it of the same degree. This last step makes the two expressions correspond term by term.

[6] For example, I say that the first equation found in the present discussion, namely

$$y^2 + \frac{qry - 2qvy + qv^2 - qs^2}{q - r},$$

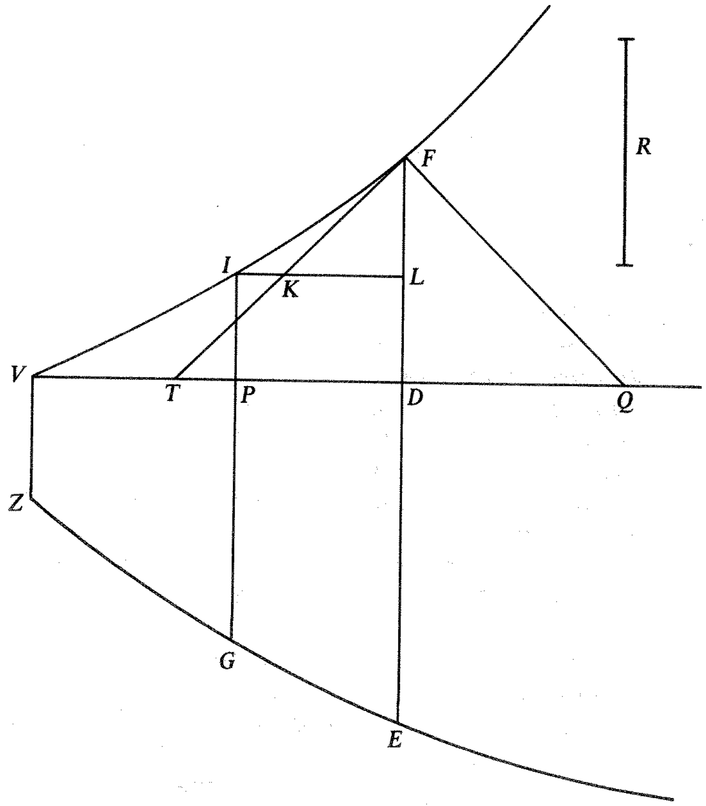
must be of the same form as the expression obtained by making $e = y$ and multiplying $y - e$ by itself, that is, as $y^2 - 2ey + e^2$. We may then compare the two expressions term by term, thus: Since the first term, y^2, is the same in each, the second term, $\dfrac{qry - 2qvy}{q - r}$, of the first is equal to $-2ey$, the second term of the second; whence, solving for v, or PA, we have $v = e - \dfrac{r}{q}e + \dfrac{1}{2}r$; or, since we have assumed e equal to y, $v = y - \dfrac{r}{q}y + \dfrac{1}{2}r$. In the same way, we can find s from the third term, $e^2 = \dfrac{qv^2 - qs^2}{q - r}$; but since v completely determines P, which is all that is required, it is not necessary to go further.

Kilde 4

11.E3 Isaac Barrow on areas and tangents

Let ZGE be any curve of which the axis is VD and let there be perpendicular ordinates to this axis (VZ, PG, DE) continually increasing from the initial ordinate VZ; also let VIF be a line such that, if any straight line EDF is drawn perpendicular to VD, cutting

the curves in the points E, F, and VD in D, the rectangle contained by DF and a given length R is equal to the intercepted space $VDEZ$; also let $DE{:}DF = R{:}DT$, and join [T and F]. Then TF will touch the curve VIF. For, if any point I is taken in the line VIF (first on the side of F towards V), and if through it IG is drawn parallel to VZ, and IL is parallel to VD, cutting the given lines as shown in the figure; then $LF{:}LK = DF{:}DT = DE{:}R$, or $R \times LF = LK \times DE$.



But, from the stated nature of the lines DF, LK, we have $R \times LF =$ area $PDEG$: therefore $LK \times DE =$ area $PDEG < DP \times DE$; hence $LK < DP < LI$.

Again, if the point I is taken on the other side of F, and the same construction is made as before, plainly it can be easily shown that $LK > DP > LI$.

From which it is quite clear that the whole of the line TKF lies within or below the curve $VIFI$.

Other things remaining the same, if the ordinates, VZ, PG, DE, continually decrease, the same conclusion is attained by similar argument; only one distinction occurs, namely, in this case, contrary to the other, the curve VIF is concave to the axis VD.